2-D Analytical Subdomain Model for High-Speed Permanent-Magnet Machines

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Abstract—This paper presents a subdomain model suitable for high-speed surface-mounted permanent-magnet machines. The model is presented with a focus on computation time and calculations of some relevant values particular to high-speed machines. The results are compared to both finite element analysis simulations and measurements on an existing prototype. The good accuracy and the high computation speed of the presented model allow an efficient optimization process. Finally, the results are discussed and further development are proposed.

Index Terms—Analytical model, eddy-current, permanent-magnet machine, slotted stator, subdomain model

I. INTRODUCTION

High-speed permanent magnet (PM) machines are increasingly used in a wide range of applications such as machine tools, gas turbine and compressors\cite{1,2,3}. Their important power density and high reliability have been found to be very suitable for high-end applications. Therefore, the need for fast and precise simulation tools has become more and more relevant. Analytical models are an efficient way to answer this problem. The computation time can be greatly reduced compared to finite element analysis (FEA) simulations and they provide a deep insight into the physics. In that category, subdomain models are particularly accurate\cite{4,5,6,7,8} since they can take into account the slotting effect and the eddy-current reaction field.

The aim of this work is to give a simple but sufficiently accurate model to compute the magnetic field in high-speed machines. The presented model applies to slotted stators without tooth-tips, single layer or double layer windings can be used and the PMs are mounted on the surface of the rotor. Two aspects are particularly investigated in that paper because of their relevance in the field of high-speed machines:

- Eddy currents induced in the PM by asynchronous harmonics can lead to critical loss, overheating and possible demagnetization\cite{8}. The suggested approach to evaluate the importance of this phenomenon is to solve the problem neglecting the time dependent term of the vector potential and then use the results to compute eddy currents in the PM. This process enables a simpler model and faster computation.
- Magnetic pressure which can cause vibrations and noise in the machine\cite{9}. The proposed method is to use the Maxwell stress tensor and the results of the model to compute the pressure.

The problem description, the assumptions and the solving method are presented in Section II. Section III presents the analytical model results and compare them to FEA simulations in terms of accuracy and computation speed. In addition, a comparison to back-EMF measurements is also made on an existing prototype. All the equations needed to program the model are given in the paper and its appendices.

II. MODEL PRESENTATION AND ANALYTICAL SOLUTION

A. Assumptions and Geometry

The following assumptions are made before simplify the model:

- infinite permeability in the rotor yoke and the stator;
- No conductivity in the rotor yoke and the stator;
- the eddy current reaction field is neglected in this model (resistance limited approximation);
- the teeth are spoke shaped (radial slot boundaries);
- no end effect;
- the PM is ring shaped (no segmentation);
- linear magnet properties.

Fig. 1 shows the main geometrical parameters of the machine. The angle $\theta_i$ stands for the angular position of the slot $i$ and $\theta_0$ is the slot opening. The number of slots will be denoted by $Q_s$.
B. Equations

From Maxwell’s equations, we can write in domain \( k \) the differential equation (1).

\[
\nabla^2 A^{(k)} = -\nabla \times B^{(k)} - \mu_0 J^{(k)} \tag{1}
\]

where \( A^{(k)} \) is the magnetic vector potential, \( B^{(k)} \) the remanent flux density, \( J^{(k)} \) the imposed current density and \( \mu_0 \) the vacuum permeability. From (1), in each subdomain, the magnetic vector potential can be more simply given by the differential equations (2) in the magnet region, (3) in the airgap and (4) in the slot regions [4], [7].

\[
\nabla^2 A^{(1)} = -\frac{\mu_0}{r} \left( M_\theta - \frac{\partial M_r}{\partial \theta} \right) \tag{2}
\]

\[
\nabla^2 A^{(2)} = 0 \tag{3}
\]

\[
\nabla^2 A^{(3i)} = -\mu_0 J_i \tag{4}
\]

where \( A^{(k)} \) is the vector potential along the \( z \)-axis, \( M_r \) and \( M_\theta \) respectively the radial and tangential magnetization (\( B = \mu_0 M \)) and \( J_i \) the current density along the \( z \)-axis in the \( i \)-th slot. \( A^{(3i)} \) varying from \( A^{(31)} \) for the first slot to \( A^{(3n)} \) for the \( Q_s \)-th slot. The general solution of (3) can be expressed as Fourier series and is given by (5).

\[
A(r, \theta) = A_0 + B_0 \ln(r) + \sum_{\lambda} \left[ \left( C_\lambda r^\lambda + E_\lambda r^{-\lambda} \right) \cos \lambda \theta \right] + \left[ \left( D_\lambda r^\lambda + F_\lambda r^{-\lambda} \right) \sin \lambda \theta \right] \tag{5}
\]

where \( A_0 \) and \( B_0 \) are scalar constants and \( C_\lambda, D_\lambda, E_\lambda \) and \( F_\lambda \) are \( \lambda \) dependent only. These coefficients will be found by solving the boundary conditions problem. The same form is used to solve (2) and (4) but a particular solution has to be added in these cases.

C. Boundary Conditions

The required boundary conditions are applied to obtain the coefficients. The first condition to be verified is the continuity of the vector potential:

\[
A^{(k)} = A^{(k+1)} \tag{6}
\]

The second condition is the continuity of the tangential magnetic field:

\[
\begin{align*}
H_r^{(k)} &= H_r^{(k+1)}, & \text{for radial boundaries} \\
H_\theta^{(k)} &= H_\theta^{(k+1)}, & \text{for circular boundaries}
\end{align*} \tag{7}
\]

Equation (7) can be used because the presented model doesn’t have any line currents. Then, the vector potential \( A \) is linked to the flux density \( B \) using (8).

\[
B = \nabla \times A \tag{8}
\]

In our case, the curl of \( A \) is more explicitly related to \( B \) by: \( B_r = \frac{1}{r} \frac{\partial A_\theta}{\partial \phi} \) and \( B_\theta = -\frac{\partial A_r}{\partial \phi} \). The field vectors \( B \) and \( H \) are coupled by:

\[
B = \mu_0 (\mu_r H + M) \tag{9}
\]

Consequently, the following boundary conditions have to be fulfilled:

- In \( R_r \), \( \forall \theta \):
  \[
  H_\theta^{(1)} \bigg|_{R_r} = 0 \tag{10}
  \]

- In \( R_m \), \( \forall \theta \):
  \[
  A^{(1)} \bigg|_{R_m} = A^{(2)} \bigg|_{R_m} \tag{11}
  \]
  \[
  H_\theta^{(1)} \bigg|_{R_m} = H_\theta^{(2)} \bigg|_{R_m} \tag{12}
  \]

- In \( R_s, \theta \in [\theta_i, \theta_i + \theta_0] \):
  \[
  H_\theta^{(2)} \bigg|_{R_s} = H_\theta^{(3i)} \bigg|_{R_s} \tag{13}
  \]
  \[
  A^{(2)} \bigg|_{R_s} = A^{(3i)} \bigg|_{R_s} \tag{14}
  \]

And otherwise:

- In \( R_{sy} \), \( \theta \in [\theta_i, \theta_i + \theta_0] \):
  \[
  H_\theta^{(3i)} \bigg|_{R_{sy}} = 0 \tag{15}
  \]

- In \( \theta_i \) and \( \theta_i + \theta_0, r \in [R_s, R_{sy}] \):
  \[
  H_r^{(3i)} \bigg|_{\theta_i} = 0 \tag{16}
  \]
  \[
  H_r^{(3i)} \bigg|_{\theta_i + \theta_0} = 0 \tag{17}
  \]

In the following subsections, the solutions for each subdomain are given. The presented form of \( A^{(1)}, A^{(2)} \) and \( A^{(3i)} \) already takes into account the boundary conditions (10), (15), (17) and (18). For that reason, the presented solutions are simpler and have fewer coefficients than in (5).

D. Magnet Region

In the magnet subdomain, the equation (2) has to be solved. The solution is a combination of a general solution and a particular solution and it’s given by (19).

\[
A^{(1)}(r, \theta) = \sum_{\lambda_1} \left[ \left( C_{1\lambda_1} (r) + \mu_0 X_{1\lambda_1} (r) \right) \cos \lambda_1 \theta \right] + \left( D_{1\lambda_1} (r) + \mu_0 Y_{1\lambda_1} (r) \right) \sin \lambda_1 \theta \tag{19}
\]

where \( C_{1\lambda_1} (r), D_{1\lambda_1} (r), X_{1\lambda_1} (r) \) and \( Y_{1\lambda_1} (r) \) are given in the Appendix and \( \lambda_1 \) is an integer varying from \( 1 \) to infinity.

E. Airgap Region

In the airgap subdomain, the equation (3) has to be solved and the solution is given by (20).

\[
A^{(2)}(r, \theta) = -\sum_{\lambda_2} \left[ \left( C_{2\lambda_2} (r) + E_{2\lambda_2} (r) \right) \cos \lambda_2 \theta \right] + \left( D_{2\lambda_2} (r) + F_{2\lambda_2} (r) \right) \sin \lambda_2 \theta \tag{20}
\]

where \( C_{2\lambda_2} (r), D_{2\lambda_2} (r), E_{2\lambda_2} (r) \) and \( F_{2\lambda_2} (r) \) are given in the Appendix and \( \lambda_2 \) is an integer varying from \( 1 \) to infinity.
F. Slot Region

In the slot subdomain, the equation (4) has to be solved and the solution is given by:

\[ A^{(s)} (r, \theta) = A3i_0 (r) + \sum_{n=1}^{N_n} E3n \lambda_n (r) \cos \lambda_n (\theta) \]  

where \( A3i_0 (r) \), \( E3n \lambda_n (r) \) and \( Z\lambda_n (r) \) are given in the Appendix and \( \lambda_n = n3 \frac{2}{\pi} \) with \( n_3 \) an integer varying from 1 to infinity.

G. Sources

The different magnetization are defined according to [4], [7]. Several definitions are allowed in this model such as sine, radial or parallel magnetizations. Concerning the current definition, the method presented in [7] is adopted. The winding can be made of a single layer or double layered non-overlapping.

H. Linear System Solving

Writing the boundary conditions lead to a linear system described by:

\[ Q \cdot X = S \]  

where \( Q \) is the boundary condition matrix, \( X \) the unknown coefficients vector and \( S \) the source vector. \( X \) and \( S \) vectors are shown in the Appendix and the boundary condition matrix is defined in 4 blocks by (23). This matrix has only 1s on its diagonal and a large amount of 0 which make it easier to invert using sparse matrices algorithms.

\[ Q = \begin{bmatrix} A & B \\ C & I \end{bmatrix} \]  

where \( I \) is the identity matrix and \( A \), \( B \), and \( C \) are defined by (46) – (??) in the Appendix.

III. RESULTS

A 6-slot 4-pole machine with concentrated windings is used to confront the model to both FEA simulations and measurements. For the model, the number of harmonics for the calculation is set to \( N_1 = 100 \) in the magnet region, \( N_2 = 100 \) in the airgap and \( N_3 = 25 \) in the slots. Concerning the FEA simulation, the mesh has around 8000 surfaces elements and at least 3 layers of elements in the airgap and in the skin depth of the PM. The geometry is identical to the analytical model and the relative permeability of the stator is set to 10000. The other parameters (current in the winding, material of the PM) are the same in both simulations.

A. Computation Time

The model is programmed using Python language with the 3.4 version. Scientific packages such as NumPy (matrices and large arrays handling) and SciPy (mathematical tools and linear algebra) are used to enable a fast and efficient computing. The computer used is a 4-core CPU Intel (R) Core (TM) i7-4600U and 8 Gb of RAM memory. The program uses matrices and vectorized functions as much as possible to accelerate the computation since the use of many ‘for’ or ‘while’ loops would dramatically slow down the program, especially in Python language. The main task of the program is to build the boundary condition matrix \( Q \) and the source vector \( S \). Another time consuming work is to inverse the linear system to obtain the solutions for each time steps. Considering the number of harmonics in each subdomain and the number of coefficients, the size of \( Q \) can be calculated using the following equation:

\[ N_Q = 2N_1 + 4N_2 + Q_s (N_3 + 1) \]  

The computation time comparison is made using 96 time steps over a half rotation of the rotor with a calculation of the current density induced in the PM. It’s difficult to strictly compare the time since it depends on many factors such as the meshing density for the FEA and the programming effectiveness as well as the number of harmonics for the analytical model. However, since the results are significantly different, we can assume that the model is much faster. Indeed, the FEA needs about 2 min whereas the model need about 0.3 second which makes it 400 times faster and shows that the program is much more efficient.

B. Flux Density, Torque and Back-EMF

In this section, comparative figures are displayed and the error of the model compared to both FEA and measured is always below 2%.

C. Magnet Eddy Current

The PM eddy current calculation is relevant for high speed machine since it gave the input to evaluate the rotor losses.
In the case of concentrated winding and large slot opening designs, these losses can be significant in the PM and cause overheating and demagnetization. The calculation of the current density in the magnet region is done using the vector potential calculated in the stationary problem. As described in [7], the current density can be expressed by:

$$J(r, \theta, t) = -\sigma \frac{\partial A}{\partial t} + C(t)$$

(25)

where $\sigma$ is the electrical conductivity of the magnet and $t$ is the time. In the case of a single ring of magnet without any segmentation, the constant $C(t)$ is found to be zero. The solution is developed using (19) and changing the reference frame with: $\theta = \theta_r + \omega_r t$.

$$J(r, \theta, t) = -\sigma R_m P_{\lambda_1} \left( \frac{R_m}{R_r} \right)$$

(26)

$$\times \sum_{\lambda_1} \left[ \frac{\partial C_1}{\partial t} + \lambda_1 \omega_r D_1 \cos \lambda_1 (\theta_r + \omega_r t) \right]$$

$$X_{\lambda_1}(r) \text{ and } Y_{\lambda_1}(r) \text{ don’t appear in this equation since they are constant in the rotor reference frame [7]. The results are shown for two different rotating speeds. Fig. (8) and (9)
show a very good agreement with the FEA in terms of both distribution and intensity of the eddy currents. Regarding the higher speed simulation presented in Fig. (10) and (11), even if the intensity of the currents if quite well predicted, the distribution differs slightly because of the resistance limited approximation.

Figure 8. Eddy currents in the PM @20 krpm (model).

Figure 9. Eddy currents in the PM @20 krpm (FEA).

Figure 10. Eddy currents in the PM @80 krpm (model).

Figure 11. Eddy currents in the PM @20 krpm (FEA).

D. Magnetic Pressure

According to Maxwell tensor and presented in [9], the radial and tangential pressure are calculated using (27) and (28). The good accuracy of the results is shown in Fig. (12).

\[
P_r(\theta) = \frac{1}{2} \left( \frac{1}{\mu_0} B_r^2 - \mu_0 H_\theta^2 \right) \quad (27)
\]

\[
P_\theta(\theta) = B_r \cdot H_\theta \quad (28)
\]

IV. Conclusion

In this paper, an analytical model dedicated to the calculation of high-speed machines was presented. The proposed method takes into account the slotting effect and can be used with surface-mounted PM with various magnetizations. In addition, the winding can be single or double layer non-overlapping. The computation speed of this model is a hundred times faster than the FEA and the accuracy is assumed to be excellent (errors are below 2% in the chosen example). Concerning the calculation of the eddy currents in the PM, the suggested method is to solve the stationary problem which is simpler and then use the results to compute the current density in the rotor. This technique allows a very fast estimation of the eddy currents but can leads to imprecisions at high rotational speeds.
For further research, the proposed subjects are: improve the calculation speed by using the periodicity of the machine; include the PM segmentation and the loss calculation in the rotor; investigate the influence of a conducting sleeve around the PM.

**APPENDIX A**

**FOURIER SERIES COEFFICIENTS**

**A. Magnet Region**

The coefficients are $C1_{\lambda_1}$ and $D1_{\lambda_1}$.

\[
C1_{\lambda_1} (r) = C1_{\lambda_1} R_m \frac{P_{\lambda_1} (r, R_r)}{P_{\lambda_1} (R_m, R_r)}
\]

\[
D1_{\lambda_1} (r) = D1_{\lambda_1} R_m \frac{P_{\lambda_1} (r, R_r)}{P_{\lambda_1} (R_m, R_r)}
\]

\[
X_{\lambda_1} (r) = \left( h_{\lambda_1} (R_r) M c_{\lambda_1} + M d_{\theta_{\lambda_1}} \right) \times \left( \frac{R_r}{R_m} \right)^{\lambda_1} \frac{P_{\lambda_1} (r, R_r)}{P_{\lambda_1} (R_m, R_r)}
\]

\[
Y_{\lambda_1} (r) = \left( h_{\lambda_1} (R_r) M s_{\lambda_1} + M s_{\theta_{\lambda_1}} \right) \times \left( \frac{R_r}{R_m} \right)^{\lambda_1} \frac{P_{\lambda_1} (r, R_r)}{P_{\lambda_1} (R_m, R_r)}
\]

with:

\[ f_{\lambda} (r) = \begin{cases} 
      \frac{1}{2} r \ln (r), & \text{if } \lambda = 1 \\
      \frac{1}{r} \ln (r), & \text{otherwise} 
   \end{cases} \]  

\[ g_{\lambda} (r) = \begin{cases} 
      \frac{1}{2} (\ln (r) + 1), & \text{if } \lambda \neq 1 \\
      \frac{1}{r} \ln (r), & \text{otherwise} 
   \end{cases} \]

\[ M c_{\lambda_1} = -M d_{\theta_{\lambda_1}} + \lambda_1 M s r_{\lambda_1} \]

\[ M s_{\lambda_1} = -M s_{\theta_{\lambda_1}} - \lambda_1 M c r_{\lambda_1} \]

where $M c r_{\lambda_1}$, $M s r_{\lambda_1}$ and $M d_{\theta_{\lambda_1}}$, $M s_{\theta_{\lambda_1}}$ respectively the radial and tangential coefficients of the magnetization in Fourier series. The functions $E_{\lambda}$ and $P_{\lambda}$ are defined in [6].

**B. Airgap Region**

The coefficients are $C2_{\lambda_2}$, $D2_{\lambda_2}$, $E2_{\lambda_2}$ and $F2_{\lambda_2}$.

\[
C2_{\lambda_2} (r) = C2_{\lambda_2} R_m \frac{P_{\lambda_2} (r, R_m)}{E_{\lambda_2} (R_r, R_m)}
\]

\[
D2_{\lambda_2} (r) = D2_{\lambda_2} R_m \frac{P_{\lambda_2} (r, R_m)}{E_{\lambda_2} (R_r, R_m)}
\]

\[
E2_{\lambda_2} (r) = E2_{\lambda_2} R_m \frac{P_{\lambda_2} (r, R_m)}{E_{\lambda_2} (R_r, R_m)}
\]

\[
F2_{\lambda_2} (r) = F2_{\lambda_2} R_m \frac{P_{\lambda_2} (r, R_m)}{E_{\lambda_2} (R_r, R_m)}
\]

**C. Slot Region**

The coefficients are $A3i_{0}$ and $E3i_{\lambda_3}$.

\[
A3i_{0} (r) = \frac{A3i_{0} R_s}{\frac{1}{2} \mu_0 j_{i_0} \left( R_s^2 \ln (r) - \frac{1}{2} r^2 \right)}
\]

\[
E3i_{\lambda_3} (r) = \frac{E3i_{\lambda_3} R_s P_{\lambda_3} (r, R_s)}{\frac{\mu_0 j_{i_{\lambda_3}}}{\lambda_3^2 - 4} R_s^2}
\]

\[
Z_{\lambda_3} (r) = \frac{\mu_0 j_{i_{\lambda_3}}}{\lambda_3^2 - 4} R_s^2 \left[ \left( \frac{r}{R_s} \right)^2 - \frac{2}{\lambda_3} \left( \frac{r}{R_s} \right)^{\lambda_3} \right]
\]

**APPENDIX B**

**MATRICES OF THE LINEAR SYSTEM**

**A. Coefficients and Source Vector**

$X$ and $S$ are column vectors defined as:

\[
X = \begin{bmatrix} C_1 & D_1 & E_2 & F_2 & C_2 & D_2 & E_{31} & \cdots & E_{3n} & A_{31} & \cdots & A_{3n} \end{bmatrix}^T
\]

\[
S = \begin{bmatrix} 0 & S_{m1} & S_{m2} & S_{j1} & S_{j2} & \cdots & S_{jm1} & S_{j01} & \cdots & S_{j0n} \end{bmatrix}^T
\]

**B. Boundary Condition Matrix**

\[
A = \begin{bmatrix} I & 0 & Q_1 & 0 & Q_5 & 0 \\
0 & I & 0 & Q_2 & 0 & Q_6 \\
0 & Q_4 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \end{bmatrix}
\]

\[
C = \begin{bmatrix} 0 & 0 & Q_{91} & Q_{101} & Q_{111} & Q_{121} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & Q_{9n_1} & Q_{10n_1} & Q_{11n_1} & Q_{12n_1} \\
0 & 0 & Q_{131} & Q_{141} & Q_{151} & Q_{161} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & Q_{13n} & Q_{14n} & Q_{15n} & Q_{16n} \end{bmatrix}
\]

where the $Q$ matrices are given in the following subsection.
C. Submatrices List

Functions $F_c$, $F_s$, $F_{cc}$, $F_{ss}$ and $F_{cs}$ are described in [10] and aimed to calculate various integrals. In this subsection, in order to save space, the notation $Q_{1,2} = F_{cc,ss}$ (for instance) means: $Q_1 = F_{cc}$ and $Q_2 = F_{ss}$. It refers to the calculation of two integrals one with cosine and the other with sine.

\[
Q_{1,2}|_{N_1, N_2} = \frac{1}{\pi r_2} \frac{P_{\lambda_2}(R_m, R_s)}{E_{\lambda_2}(R_m, R_s)} \times F_{cc, ss}[1, 0, 0, 0, 0, 0, 2\pi]
\]

\[
Q_{3,4}|_{N_1, N_1} = \frac{\lambda_1}{\pi \mu_r} \frac{P_{\lambda_1}(R_m, R_r)}{E_{\lambda_1}(R_m, R_r)} \times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 2\pi]
\]

\[
Q_{5,6}|_{N_1, N_2} = \frac{2}{\pi r_2} \frac{E_{\lambda_2}(R_s, R_m)}{r_m/r_s}
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 2\pi]
\]

\[
Q_{7i, 8i}|_{N_2, N_3} = \frac{\lambda_3}{\pi} \frac{E_{\lambda_3}(R_s, R_{sy})}{P_{\lambda_3}(R_s, R_{sy})}
\times F_{cc, ss}[1, \lambda_2, \lambda_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

\[
Q_{9i, 10i}|_{N_3, N_2} = \frac{\theta_0 \lambda_2}{\pi r_m} \frac{E_{\lambda_2}(R_m, R_s)}{P_{\lambda_2}(R_m, R_s)}
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

\[
Q_{11i, 12i}|_{N_3, N_2} = \frac{2}{\pi r_m} \frac{E_{\lambda_2}(R_s, R_m)}{P_{\lambda_2}(R_s, R_m)}
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

\[
Q_{13i, 14i}|_{N_1, N_2} = \frac{\theta_0 \lambda_2}{\pi r_s} \frac{E_{\lambda_2}(R_s, R_r)}{P_{\lambda_2}(R_s, R_r)}
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

And for the source vector:

\[
[S_{m1,m2}|_{N_2, N_1} = \left( h_{\lambda_1}(R_s) M_c s_{\lambda_1} + M_c s_{\lambda_1} \right) \frac{P_{\lambda_1}(R_m, R_r)}{E_{\lambda_1}(R_m, R_r)} + 1 \times \left( 1 + \frac{E_{\lambda_1}(R_m, R_r)}{P_{\lambda_1}(R_m, R_r)} \right)
\]

\[
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

\[
[S_{j_{c1}, j_{s2}}|_{N_2, N_1} = \frac{-1}{2\pi} \mu_0 j_{01} \frac{1}{R_s} \left( R_{sy}^2 - R_s^2 \right)
\]

\[
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

\[
[S_{j_{11}}, N_1] = \frac{-\mu_0 j_{12} \lambda_2 R_{sy}^2}{\lambda_3 \lambda_4 - 4} \frac{E_{\lambda_3}(R_s, R_{sy})}{P_{\lambda_3}(R_s, R_{sy})}
\]

\[
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

\[
[S_{j_{10}}, N_1] = \frac{-\mu_0 j_{01} \lambda_2 R_{sy}^2}{\lambda_3 \lambda_4 - 4} \frac{E_{\lambda_3}(R_s, R_{sy})}{P_{\lambda_3}(R_s, R_{sy})}
\]

\[
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

\[
[S_{j_{01}}, N_1] = \frac{-\mu_0 j_{12} \lambda_2 R_{sy}^2}{\lambda_3 \lambda_4 - 4} \frac{E_{\lambda_3}(R_s, R_{sy})}{P_{\lambda_3}(R_s, R_{sy})}
\]

\[
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

\[
[S_{j_{00}}, N_1] = \frac{-\mu_0 j_{01} \lambda_2 R_{sy}^2}{\lambda_3 \lambda_4 - 4} \frac{E_{\lambda_3}(R_s, R_{sy})}{P_{\lambda_3}(R_s, R_{sy})}
\]

\[
\times F_{cc, ss}[1, \lambda_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2\pi]
\]

REFERENCES